Numerical modeling in elastic solid and viscoplastic fluid

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Plan

1. Modeling of elastic solids
2. Modeling of viscoplastic flows

Plan

Introduction

Frame: foam.
Behavior: both solid or liquid.
Features: light, large space, insulating.
Motivation: insulates, infiltrates pores, maintain debris in suspension...

Multi-scale:

- macro: solid,
- milli: bubbles,
- micro: films,
- nano: molecule.

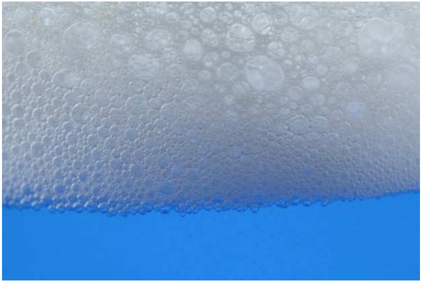


Figure : Soap foam.

Focus

- Small liquid fraction.
- Foam at steady state.
- Solid behavior.

- Scale of the bubble.
- Homogeneization.

- Mechanical characteristic.
- Thermal conductivity.

Geometry

Foam

Liquid fraction $< 6,3\%$ => ordered stack of **bubble** with **Kelvin** geometry.

Bubble: polyhedron made of films (fluid) containing gas (air).

Kelvin cell:

- 8 hexagonal faces,
- 6 square faces,
- 36 edges of length l.

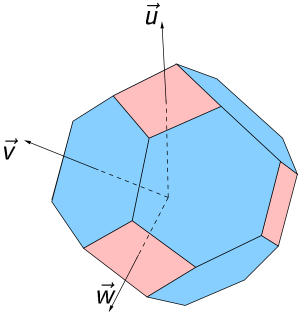


Figure : Kelvin cell.

Rheology

Type of behavior

- **Elastic**: foam deforms reversibly.
- **Plastic**: threshold beyond which deformation is irreversible (rearrangements of bubbles).
- **Viscous**: foam flows as a liquid.

Elasticity

Linear static behavior of an elastic solid (**Hooke's law**):

$$\sigma = G\varepsilon,$$

where G the elastic modulus,
σ the stress tensor,
ε the strain tensor.

Plan

Linear elasticity

Domain Ω of dimension 3, point $\vec{x} = (x, y, z) \in \Omega$.
Displacement $\boldsymbol{u}(\vec{x}) = (u_x, u_y, u_z)(\vec{x})$.

- Small deformation => linear strain tensor

$$\varepsilon(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t\right) \quad \text{symetric.}$$

- Isotropic.
- Constitutive equation: (**Hooke's law**)

$$\sigma = 2\mu\varepsilon + \lambda \mathrm{tr}\varepsilon \mathbb{1} \quad \text{symetric.}$$

with
  - σ Cauchy stress tensor.
  - μ and λ Lamé coefficients.
- Steady state

$$-\operatorname{div}\sigma = 0 \quad \text{in } \Omega.$$

Resolution

- **Week formulation:**
  existence and uniqueness of the solution.

- **Galerkin method:**
  approached problem => approached solution (discrete).

- **Finite element:**
  - decomposition of the domain in finite element,
  - construction of a space of approximation,
  - assembly of the stiffness matrix,
  - resolution of the associated linear system.

Formulation

Divergence theorem

Let $\Omega \subset \mathbb{R}^n$, if $u \in \mathcal{C}^1(\Omega)$ then

$$\int_{\boldsymbol{\Omega}} \mathrm{div}\, u \, \boldsymbol{dV} = \int_{\boldsymbol{\partial\Omega}} u \cdot \vec{N} \, \boldsymbol{dS}$$

where $\vec{N}$ is unit normal vector.

Variational formulation

Let $V = (H^1_{bc}(\Omega))^3$, we seek for $u \in V$ such that

$$\int_{\Omega} \sigma(u) : \varepsilon(v) - \int_{\partial\Omega} (\sigma(u) \cdot \vec{N}) \cdot v = 0, \qquad \forall v \in V.$$

where $\sigma \cdot \vec{N}$ normal stress vector.

Boundary conditions: $\partial\Omega = \boldsymbol{\Gamma_u} \cup \boldsymbol{\Gamma_\sigma}$

- $u = u_d$ on $\boldsymbol{\Gamma_u}$ imposed displacement.
- $\sigma \cdot \vec{N} = T_d$ on $\boldsymbol{\Gamma_\sigma}$ tension effect.

Discretization

- **Galerkin:** consider $V_h \subset V$ of **finite** dimension N:
  - base: fonctions $(\varphi_i)_{i=1}^N$,
  - $u_h(x) = \sum_{i=1}^{N} \varphi_i(x) u_i$.

- Approched problem **(VF)**$_h$:
  seek for $u_h \in V_h$ such that

$$\int_\Omega \sigma(u_h) : \varepsilon(v_h) - \int_{\partial\Omega} (\sigma(u_h) \cdot \vec{N}) \cdot v_h = 0, \qquad \forall v_h \in V_h.$$

- **(VF)**$_h$ $\Leftrightarrow$ linear system $A_h u_h - F_h = 0$.

Finite element method

- **Mesh Ω_h:**
  - $\overline{\Omega} = \bigcup_{K \in \Omega_h} K$,
  - K polygon/polyhedron,
  - h max diameter of elements.

- **Space V_h:**
  - continuous,
  - polynomial by piece,
  - small support.

- Stiffness matrix A_h.

- Solve $A_h u_h = F_h$.

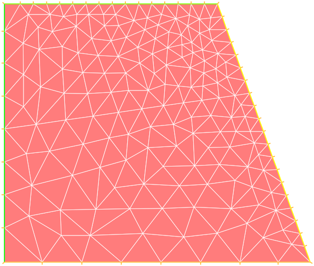


Figure : Triangular mesh.

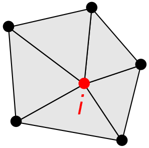


Figure : Support of φ_i.

Homogeneization

- **Foam**:
  - **microscopic** level,
  - **heterogeneous** medium.

- **Representative elementary volume**:
  - **macroscopic** level,
  - **homogeneous** and continuous medium.

Representative elementary volume (REV)

l: size of the (REV),
d: size of heterogeneities,
L: size of the structure,

- ($l \gg d$) $\Rightarrow$ (REV) represente the heterogeneous microstructure.
- ($l \ll L$) $\Rightarrow$ the structure is a continuous medium.

Methodology

- Solve a boundary value problem on the basic cell $\mathcal{V}$:
  - appropriate boundary conditions,
  - elementary loading (traction, shearing).

- Extract the effective mechanical properties:
  micro deformation: $< \varepsilon >$,
  micro constraint: $< \sigma >$.

- Deduce the heterogeneous behavior:
  macro deformation: $\boldsymbol{E} =< \varepsilon >$,
  macro constraint: $\boldsymbol{\Sigma} =< \sigma >$.

Problem formulation 1/3

- **Basic cell:** $\mathcal{V}$ = Kelvin cell.
- 3 symmetry planes $\Longrightarrow$
- **Periodic** boundary conditions.

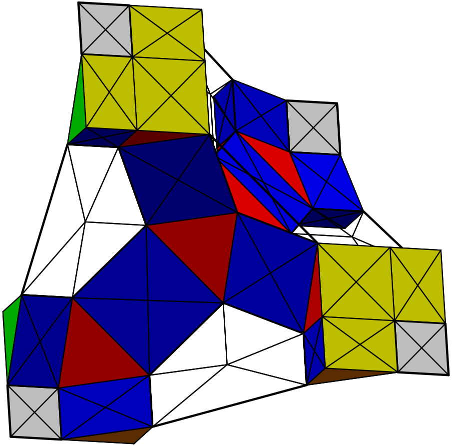


Figure : $\frac{1}{8}$ of Kelvin cell.

Problem formulation 2/3

- Heterogeneous medium:
  ε oscillates around its mean value $< \varepsilon >$ noted $\boldsymbol{E}$.

- Decomposition of the displacement

$$u(\vec{x}) = \boldsymbol{E} \cdot \vec{x} + \overline{\boldsymbol{u}}(\vec{x}) \quad \forall \vec{x} \in \mathcal{V},$$

  where $\overline{\boldsymbol{u}}$ is **periodic**.

- Deformation

$$\varepsilon(u(\vec{x})) = \boldsymbol{E} + \varepsilon(\overline{\boldsymbol{u}}(\vec{x})) \quad \forall \vec{x} \in \mathcal{V}.$$

  $\varepsilon(\overline{\boldsymbol{u}}(\vec{x}))$: fluctuating term (heterogeneities).

Problem formulation 3/3

Local problem on $\mathcal{V}$:

$$
\begin{aligned}
&-\operatorname{div}\sigma(\vec{x}) = 0 \quad \forall \vec{x} \in \mathcal{V},\\
&\sigma = 2\mu\varepsilon(u) + \lambda \mathrm{tr}\varepsilon(u)\mathbb{1},\\
&\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^t),\\
&u(\vec{x}) = \boldsymbol{E}\cdot\vec{x} + \overline{\boldsymbol{u}}(\vec{x}), \quad \forall \vec{x} \in \mathcal{V},\\
&\overline{\boldsymbol{u}} = \overline{\boldsymbol{u}}_d \text{ on } \Gamma_u,\\
&\sigma\cdot\vec{N} = T_d \text{ on } \Gamma_\sigma.
\end{aligned}
$$

=> local fields $\boldsymbol{\sigma}$ and ε induced by the macro deformation $\boldsymbol{E}$.

Homogenisation for periodic media

Uniqueness of the effective properties $<\boldsymbol{\sigma}>$ and $<\varepsilon>$,
calculated on the basic cell $\mathcal{V}$.

Mean value

- Apply macro deformation (3 tractions, 3 shears).
- Determination of $<\boldsymbol{\sigma}>$:
  - mean formule:

$$<\boldsymbol{\sigma}>:\boldsymbol{E} = \frac{1}{|\Omega|}\int_{\Omega}\boldsymbol{\sigma}:\boldsymbol{E}\,dV$$

  - week formulation in $v = \boldsymbol{E}\cdot\vec{x}$

$$<\boldsymbol{\sigma}>:\boldsymbol{E} = \frac{1}{|\Omega|}\int_{\partial\Omega} T\cdot(\boldsymbol{E}\cdot\vec{x})\,dS$$

Example

$\boldsymbol{E} = \boldsymbol{e_x}\otimes\boldsymbol{e_x}$ traction in x direction:

$$<\sigma_{xx}> = \frac{1}{|\Omega|}\int_{\partial\Omega} T\cdot\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} dS$$

Elasticity tensor

- Generalized Hooke's law:

$$\sigma = \mathcal{C} : \varepsilon \quad \text{that is} \quad \sigma_{ij} = \mathcal{C}_{ijkl}\varepsilon_{kl},$$

with $\mathcal{C}$ the elasticity tensor (symmetric).

- Equivalent homogeneous law:

$$\begin{array}{ccccc} \boldsymbol{\Sigma} & = \mathbb{C} : & \boldsymbol{E}, \\ \parallel & & \parallel \\ <\boldsymbol{\sigma}> & & <\boldsymbol{\varepsilon}> \end{array}$$

with $\mathbb{C}$ the macro elasticity tensor.

- Voigt notation:

$$\begin{pmatrix} \Sigma_{xx} \\ \Sigma_{yy} \\ \Sigma_{zz} \\ \Sigma_{xy} \\ \Sigma_{yz} \\ \Sigma_{xz} \end{pmatrix} = \begin{pmatrix} \mathbb{C}_{xxxx} & \mathbb{C}_{xxyy} & \mathbb{C}_{xxzz} & \mathbb{C}_{xxxy} & \mathbb{C}_{xxyz} & \mathbb{C}_{xxxz} \\ x & \mathbb{C}_{yyyy} & \mathbb{C}_{yyzz} & \mathbb{C}_{yyxy} & \mathbb{C}_{yyyz} & \mathbb{C}_{yyxz} \\ x & x & \mathbb{C}_{zzzz} & \mathbb{C}_{zzxy} & \mathbb{C}_{zzyz} & \mathbb{C}_{zzxz} \\ x & x & x & \mathbb{C}_{xyxy} & \mathbb{C}_{xyyz} & \mathbb{C}_{xyxz} \\ x & x & x & x & \mathbb{C}_{yzyz} & \mathbb{C}_{yzxz} \\ x & x & x & x & x & \mathbb{C}_{xzxz} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz}. \end{pmatrix}$$

Thermal conduction

Domain Ω of dimension 3, point $\vec{x} = (x, y, z) \in \Omega$.
Temperature $T(\vec{x})$, heat flow $\vec{q}(\vec{x})$.

- Conduction: transfer of energy between particles.
- Constitutive equation: (**Fourier's law**)

$$\vec{q} = -\kappa \nabla T \quad \text{in } \Omega.$$

  where κ is the conductivity coefficient.
- Thermal balance law

$$\mathrm{div}\, \vec{q} = 0 \quad \text{in } \Omega.$$

Variational formulation

We seek for $T \in H^1_{bc}(\Omega)$ such that

$$\int_\Omega \kappa \nabla T \cdot \nabla v - \int_{\partial\Omega} \kappa (\nabla T \cdot \vec{N}) v = 0, \quad \forall v \in H^1_{bc}(\Omega).$$

Boundary conditions: $\partial\Omega = \boldsymbol{\Gamma_T} \cup \boldsymbol{\Gamma_q}$

- $T = T_d$ on $\boldsymbol{\Gamma_T}$ imposed temperature (isotherm).
- $\vec{q} \cdot \vec{N} = 0$ on $\boldsymbol{\Gamma_q}$ no flux (insulation).

Local problem on $\mathcal{V}$ with a macro loading $\boldsymbol{A}$

$$\begin{aligned}
&\mathrm{div}\, \vec{q}(\vec{x}) = 0 \quad \forall \vec{x} \in \mathcal{V}, \\
&\vec{q} = -\kappa \nabla T \quad \text{in } \mathcal{V}, \\
&T(\vec{x}) = \boldsymbol{A} \cdot \vec{x} + \boldsymbol{\tau}(\vec{x}), \quad \forall \vec{x} \in \mathcal{V}, \ \text{with } \boldsymbol{\tau} \text{ periodic.} \\
&\boldsymbol{\tau} = \tau_d \text{ on } \Gamma_T, \\
&\vec{q} \cdot \vec{N} = 0 \text{ on } \Gamma_q.
\end{aligned}$$

Homogeneization

- Apply 3 loading $\boldsymbol{A}$: directions x, y and z.
- Determination of mean flux

$$< \vec{q} >= \frac{1}{|V|} \int_{\partial\Omega} (\vec{q} \cdot \vec{N}) \cdot \vec{x} \, dS$$

- Macroscopic flux:

$$\boldsymbol{Q} =< \vec{q} > .$$

Equivalent homogeneous law

$$\boldsymbol{Q} = \mathbb{K} \cdot \nabla T,$$

where $\mathbb{K}$ is the macro conductivity tensor.

Plan

Incompressibility

- Lamé coefficient λ linked to the compressibility of the material:

$$\lambda \to \infty \Rightarrow \textbf{incompressible} \text{ material}$$
$$(\operatorname{div} u \to 0)$$

- Replace $-\lambda \operatorname{div} u$ by p and consider

$$\sigma = 2\mu\varepsilon(u) - p\mathrm{Id} \quad \text{(constitutive equation)},$$
$$\operatorname{div} u = 0 \quad \text{(mass conservation)}.$$

- Balance

$$-\operatorname{div} \sigma = 0 \quad \text{in } \Omega.$$

+ boundary conditions.

- Two unknows $(u, p) \Rightarrow$ mixte variational formulation.
- Finite element approximation $\mathbb{P}^{\mathbf{2}}/\mathbb{P}^{\mathbf{1}}$.

Plan

- Geophysical flows modeling: avalanches, landslides, pyroclastic flows.
- Study of the fluid/solid transition in a two-phase fluid.
- Dynamics with motionless part at the bottom and mobile on the surface.

Main difficulties:

- Rheology: - viscosity and plasticity,
  - yield stress, transition.
- Free surface.

Yield stress fluid

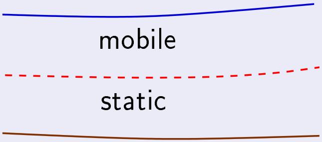
Features:

- Existence of a flow threshold.
- Below the yield limit, the stress cannot cause the flow.

Phase transition

Simultaneous existence:

static part & mobile part.



Viscoplastic rheology

- $\vec{U}$ velocity, p pressure.
- Stress tensor: $P = \sigma - p\mathrm{Id}$, $\mathrm{tr}\sigma = 0$.

Viscoplastic law ⇒ yield criterion

- Bingham law:

$$\begin{aligned} \sigma &= 2\eta D\vec{U} + \boldsymbol{\sigma_c}\frac{D\vec{U}}{||D\vec{U}||} && \text{if } D\vec{U} \neq 0, \\ \|\sigma\| &\leq \boldsymbol{\sigma_c},\ \sigma \text{ symmetric} && \text{if } D\vec{U} = 0, \end{aligned}$$

  with $D\vec{U}$ the strain tensor and η the viscosity.

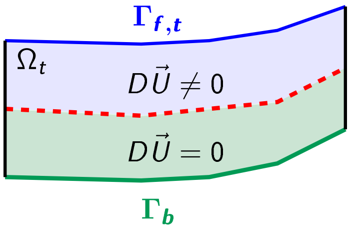
- **Drucker–Prager** law:

$$\begin{aligned} \sigma &= 2\eta D\vec{U} + \boldsymbol{\kappa(p)}\frac{D\vec{U}}{||D\vec{U}||} && \text{if } D\vec{U} \neq 0, \\ ||\sigma|| &\leq \boldsymbol{\kappa(p)},\ \sigma \text{ symmetric} && \text{if } D\vec{U} = 0, \end{aligned}$$

  with $\kappa(p)$ the plasticity.

Domain description

- Domain Ω_t time dependent (**free surface $\Gamma_{f,t}$**).
- Fixed **bottom Γ_b**.



- Boundary condition: $P.\vec{N} = 0$ on $\Gamma_{f,t}$ and $\vec{U} = 0$ on Γ_b.
- Kinematic condition: $N_t + \vec{N} \cdot \vec{U} = 0$ on $\Gamma_{f,t}$.
- Fluid/solid **interface** between two phases, characterized by $D\vec{U}$ zero or $D\vec{U}$ nonzero.
- **Interface** not described explicitly.

Conservation equations

- **Incompressible** Navier–Stokes equations:

$$\begin{aligned} \rho\left(\partial_t \vec{U} + (\vec{U}.\nabla_{\vec{X}})\vec{U}\right) + \mathrm{div}_{\vec{X}} P &= \rho\vec{f} && \text{in }]0,T[\times\Omega_t, \\ \mathrm{div}_{\vec{X}} \vec{U} &= 0 && \text{in }]0,T[\times\Omega_t. \end{aligned}$$

- **Viscoplastic** fluid of Bingham type:

$$\begin{aligned} \sigma = 2\eta D\vec{U} + \kappa(p)\frac{D\vec{U}}{\|D\vec{U}\|} && \text{if } D\vec{U} \neq 0, \\ \|\sigma\| \leq \kappa(p),\ \sigma \text{ symmetric} && \text{if } D\vec{U} = 0, \end{aligned}$$

where $D\vec{U} = \frac{\nabla\vec{U}+\nabla\vec{U}^t}{2}$.

- Drucker–Prager relation:

$$\boldsymbol{\kappa(p) = \sqrt{2}\mu[p]_+},$$

μ internal friction coefficient.

Overall presentation

One dimensional:

$$\begin{aligned} \partial_t U + \mathbf{S} - \nu \partial^2_{ZZ} U &= 0 \quad \forall Z \in]b(t), h[, \\ U &= 0 \quad \text{in } Z = b(t), \\ \nu \partial_Z U &= 0 \quad \text{in } Z = b(t), \\ \nu \partial_Z U &= 0 \quad \text{in } Z = h. \end{aligned}$$

- simplified model with source term,
- formulation with $b(t)$ the interface,
- inviscid case: analytical solution,
- viscous case: - numerical resolution
  - comparison with experiments.

Two dimensional:

...

Resolution

- **Regularization** by $\epsilon \ll 1$ of the constitutive equation.
- **ALE** method for the displacement of the domain Ω_t.
- Space discretization in $\mathbb{P}^2/\mathbb{P}^1$ finite elements.
- Numerical local surface tension (erodible bed).

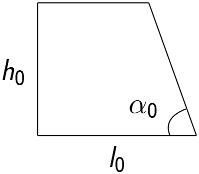
Plan

Granular collapse

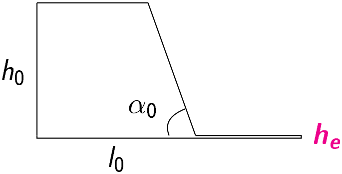
Comparison:

- Thickness profile (**free surface** evolution).
- Position of the static/flowing **interface**.

Configurations:



(a) rigid bed

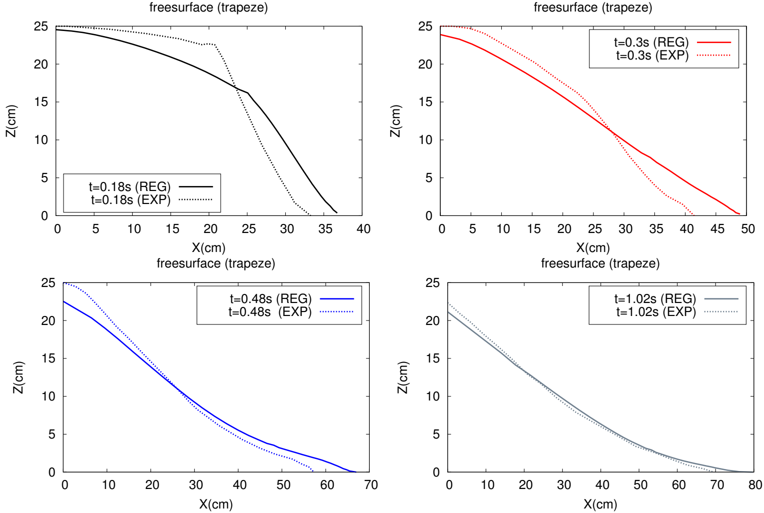


(b) erodible bed

Parameters:

$\alpha_0 = 70^\circ$, $h_0 = 25\mathrm{cm}$, $h_e = 5\mathrm{mm}$, $l_0 = 29.7\mathrm{cm}(a)/80\mathrm{cm}(b)$.

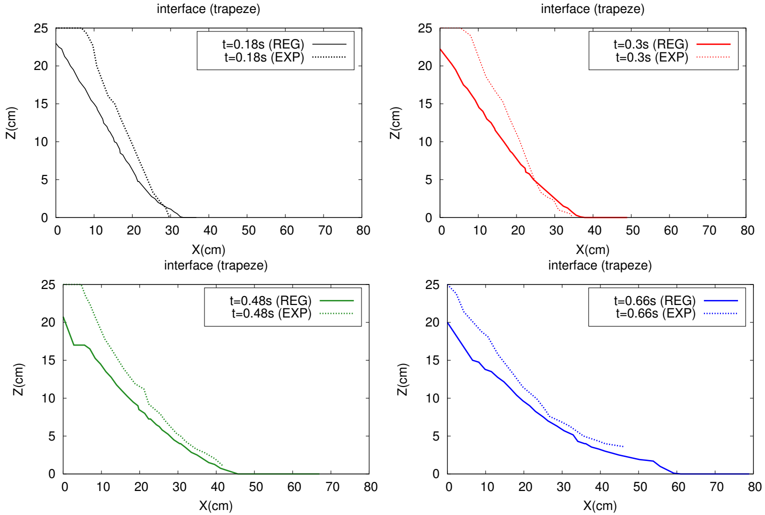
Freesurface (rigid bed)



<u>The regularization method leads to:</u>

- faster dynamics than in experiments,
- final deposit very well approximated.

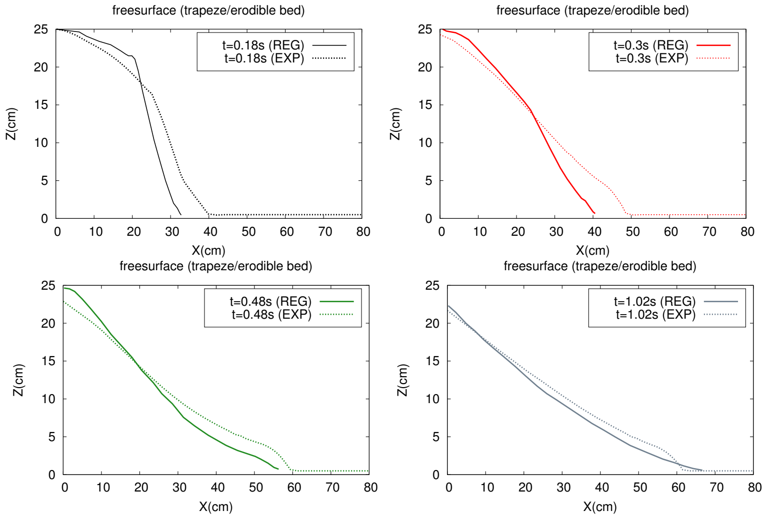
Static/flowing interface (rigid bed)



The regularization method leads to:

- overall good approximation of interface position,
- position underestimated on top of the left side.

Freesurface (erodible bed)



The regularization method with **local surface tension** leads to:

- comparable profiles all along the simulation,
- final deposit well approximated.

Conclusions (Geophysics):

- Two-dimensional flows of viscoplastic materials with pressure-dependent yield stress.
- Regularization method with evolution of the mesh.
- Granular collapse with erosion process.

Perspectives (Mecanics):

- Reinforcement by particles.
- Film thickness.
- Thermics (radiation) and acoustic.
- Study of transition fbb-bcc.